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DUAL METHODS FOR OPTIMIZING FINITE ELEMENT FLEXURAL SYSTEMS

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Modern numerical methods for the optimization of large discretized systems are now well developed and highly efficient in the case of thin walled elastic structures modeled by finite elements. However, this is not yet true for structures whose components are subject simultaneously to bending and extension loads. In this paper, the idea of Generalized Optimality Criterion (GOC), set forth in previous final scientific reports for bar, membrane and pure bending elements, is extended to deal with general beam and flat shell elements. The modifications brought to the GOC result in explicit approximations for the behavior constraints

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TABLE OF CONTENTS

SUMMARY

Modern numerical methods for the optimization of large discretized systems are now well developed and highly efficient in the case of thin walled elastic structures modeled by finite elements. However, this is not yet true for structures whose components are subject simultaneously to bending and extension loads. In this paper, the idea of Generalized Optimality Criterion (GOC), set forth in previous final scientific reports for bar, membrane and pure bending elements, is extended to deal with general beam and flat shell elements. The modifications brought to the GOC result in explicit approximations for the behavior constraints that are still correct up to the first order, but that exhibit a more complex algebraic form. Indeed these explicit expressions are no longer merely linear in the reciprocal design variables. However they continue to be additively separable and therefore, dual methods remain fully applicable, just as in the original statement of the GOC approach. Numerical examples will be offered to demonstrate the efficiency of the method presented.

1. INTRODUCTION

The optimum design of any significant structure is the result of a delicate compromise between many complex factors. Some are rational and can be quantified, such as the strength of the structure, its natural frequencies, its weight, ... Some are just as rational but are difficult to quantify, such as the experience of the designer in a given technology. Some others are much less rational, like styling, but are just as important for the final goal of the process, which is the marketing. Naturally a good designer considers structural optimization as a technique that should take into account all possible aspects of the design. Consequently the designers are often reluctant to the concepts of structural optimization developed in connection with finite element programs.

However a more detailed examination of the design process leads to isolate a phase that appears frequently, during which the shape of the structure is more or less frozen and the problem is limited to giving adequate dimensions to the various members. Such a situation is often encountered in the aerospace, naval or automobile industries, where the external shapes are, to a large extent, dictated by aero- or hydrodynamic considerations, or by styling, while internal forms are often determinated by various other non structural requirements. If the ultimate goal of the designer can be identified as corresponding to the minimization of an explicit function of the member sizes, and if the limitations on the design can be defined as, eventually implicit, functions of the member sizes too, such as displacements, stresses, eigenfrequencies, etc..., then the problem is tractable by automatic algorithms. They allow the designer to speed up significantly this part of the design process and to explore more systematically the various feasible designs.

The optimization problems, which in fact should be called automatic sizing problems, are especially crucial when complex structural forms are involved, when flexural forces cannot be

neglected, and when composite materials such as reinforced resins are employed. In these cases it becomes difficult, if not impossible, for the designer to have an intuitive understanding of the structural mechanics that is sufficient to lead to optimal sizing of the various members. Furthermore, the designer is most of the time unable to take into account global constraints in the structure, like global flexibility, restriction on displacements, frequencies of vibration, global buckling modes, etc... It is only possible to verify a posteriori that such constraints are satisfied. Again these global constraints become more important in the context of highly, indeterminate structures. In the aerospace industry, the necessity of designing high performance structures has motivated significant research efforts to derive algorithms permitting a rapid and systematic exploration of the design space to determine the optimum material utilization.

It is worth pointing out that optimization methods should be considered as especially useful in the preliminary design phase. Using them when the design is practically frozen, with the hope of an ultimate improvement, is often disappointing. This is due to the fact that the optimization of a detailed design implies the formulation of a large number of constraints, some of which are not easily quantified. At the preliminary design stage, however, the constraints are usually more global and therefore more easily handled by the available formulations.

The structural optimization problem considered in this report consists of the weight minimization of a finite element model with fixed geometry and material properties. The design variables are taken as the transverse sizes of the structural members, namely, the cross-sectional areas of bar and beam elements and the thicknesses of membrane, plate and flat shell elements. The mathematical programming problem to be solved has the following form :

$$\text{minimize} \quad W = \sum_{i=1}^{n} \ell_i \, a_i \tag{1}$$

$$\text{subject to} \quad h_j(a) \geq 0 \qquad j = 1,m \tag{2}$$

$$\bar{a}_i \geq a_i \geq \underline{a}_i \tag{3}$$

The a_i's denote the n design variables. They correspond to member sizes of either individual finite elements, or, if design variable linking is used, of groups of finite elements. The structural weight W is a linear objective function, because the ℓ_i's are constant coefficients representing the specific weight times either the element length (bars and beams) or the element area (membranes, shear panels, plates and flat shells). The inequalities (2) are the behavior constraints, which impose limitations on quantities describing the structural response, for example, the stresses and displacements under multiple loading cases, the natural frequencies, the buckling loads, etc... The design variables are also subjected to the side constraints (3), where $\underline{a}_i$ and $\bar{a}_i$ are lower and upper limits that reflect fabrication and analysis validity considerations.

Standard minimization techniques can be applied to the non-linear programming problem (1-3). However this problem exhibits some characteristics that make it complicated when practical structural design applications are considered. The essential difficulty arises from the implicit nature of the behavior constraints (2), in that their precise numerical evaluation for each particular design requires a complete finite element analysis. Since the solution scheme is iterative, it involves a large number of structural reanalyses. Therefore the computational cost often becomes prohibitive when large structural systems are dealt with. However a powerful design procedure has now emerged, which consists in replacing the initial problem with a sequence of simple explicit problems. In the next section this approach will be briefly reviewed by restricting the formulation to thin walled structures idealized by bar/membrane elements. A much more detailed presentation can be found in a previous report [1]. It will be shown that the behavior constraints can be approximated either by using virtual load considerations (optimality criteria approach) or by using first order Taylor series expansion with respect to the reciprocal design variables (mathematical programming approach). Applying a dual solution scheme to each explicit problem generated in sequence naturally introduces the concept of a generalized optimality criterion.

Subsequently sections 3 and 4 will be concerned with structural models that are capable of carrying flexural forces. For beams and plates in pure bending, adequate intermediate variables can be selected, in terms of which high quality explicit approximations for the behavior constraints can still be generated by linearization. The idea of generalized optimality criterion remains fully valid and it keeps its interpretation in terms of energy densities in the structural members. Section 3 is a summary of results presented in a previous report (ref. [2], section 6).

The essential problem is addressed in section 4. It consists of the establishment of the generalized optimality criterion approach in the general case where the structural members work both in extension and flexion (beam and flat shell elements). For displacement constraints, it is no longer possible to select a suitable intermediate variable for the linearization process, however, the virtual load procedure permits obtention of high quality, first order explicit approximations of the behavior constraints. It will also be shown how to proceed for stress, frequency and buckling constraints.

The explicit approximations still exhibit a separable algebraic form and therefore dual methods remain applicable. However some difficulties might happen due to the lack of convexity in the explicit subproblem. Section 5 will describe how to solve the explicit subproblems by using dual methods. Finally some applications will be offered in section 6 to illustrate the power and generality of the approach presented.

2. GENERALIZED OPTIMALITY CRITERION FOR THIN WALLED STRUCTURES

This section summarizes some results obtained in a previous work [1] for structural models made up of bar, membrane and shear panel elements, which are quite adequate for idealizing thin walled structures subjected mainly to extension loading. For this class of finite element models, the structural stiffness matrix exhibits a linear form in the design variables :

$$K = \sum_{i=1}^{n} K_i = \sum_{i=1}^{n} a_i \bar{K}_i \qquad (4)$$

where $\bar{K}_i$, a matrix of constants, represents the stiffness matrix of the ith element when $a_i = 1$. For simplicity, the following discussion is restricted to problems involving constraints on static stresses and displacements, in which case the behavior constraints (2) can be written

$$h_j(a) \equiv \bar{u}_j - u_j(a) \geq 0 \qquad (5)$$

where $\bar{u}_j$ denotes an upper bound to a response quantity $u_j(a)$ (stress, nodal displacement, relative displacement).

Most of the optimality criteria approaches (e.g. [3]), as well as the generalized optimality criterion (GOC) set forth in Ref. [1], use the virtual load technique to generate explicit approximations of the stress and displacement constraints. Introducing a virtual load vector conjugated to the response quantity u_j (unit load for a nodal displacement), it follows that u_j can be expressed as the sum of the contributions of each finite element :

$$u_j = q^T K q_j = \sum_{i=1}^{n} \frac{c_{ij}}{a_i} \qquad (6)$$

with

$$c_{ij} = (q^T \bar{K}_i q_j) a_i^2 \qquad (7)$$

In these expressions q and q_j are respectively the real and virtual displacement vectors and $\tilde{K}_i$ is the element stiffness matrix appearing in (4). It can be seen from (7) that the coefficients c_{ij} are related to the virtual strain energy densities in the structural members. The c_{ij}'s are constant coefficients in the case of a statically determinate structure, so that (6) represents then the exact explicit form of the response quantity u_j. In the case of a statically indeterminate structure, the c_{ij}'s depend implicitly on the design variables, because structural redundancy produces redistribution of the internal forces when the member sizes are modified. Therefore the following explicit constraints

$$\tilde{h}_j(a) \quad \bar{u}_j - \sum_{i=1}^{n} \frac{c_{ij}}{a_i} \geq 0 \tag{8}$$

constitute in general approximate forms of the original constraints (5). As shown in Ref. [1], the basic idea in the optimality criteria approach can be viewed as transforming the initial implicit problem into a sequence of explicit subproblems. Each explicit problem results from replacing the behavior constraints (2) by their approximate forms (8).

On the other hand, the mathematical programming approach to structural optimization, after a period of unefficiency, has finally evolved into a powerful and now well established design procedure which is also based upon explicit approximations of the behavior constraints [4, 5, 6]. The key idea is to linearize the behavior constraints with respect to the reciprocal design variables

$$x_i = \frac{1}{a_i} \tag{9}$$

Justification for this change of variables lies in the fact that the constraint surfaces can be shown to be very shallow and close to planes in the reciprocal design variable space. Therefore the linearized forms of the constraints are usually high quality approximations. They are obtained by using first order Taylor series expansion in terms of the reciprocal variables x_i :

$$\tilde{h}_j(x) = \bar{u}_j - [u_j^o + \sum_{i=1}^{n} \left(\frac{\partial u_j}{\partial x_i}\right)^o (x_i - x_j^o)] \geq 0 \tag{10}$$

where the upperscript ^{o} denotes quantities evaluated at the actual design point x^o, where the structural analysis is performed. Note that the finite element analysis capability must include auxiliary sensitivity analyses for evaluating the first partial derivatives of the response quantities. Most often the well known pseudo-loads technique is employed [7].

It has been shown in a previous report [1] that the explicit approximations of the behavior constraints used in both the optimality criteria and mathematical programming approaches (Eqs 8 and 10, respectively) are identical. Indeed the virtual strain energy densities c_{ij} employed in the optimality criteria approaches are nothing else that the gradients of the response quantities with respect to the reciprocal variables :

$$c_{ij} = \frac{\partial u_j}{\partial x_i} \tag{11}$$

Furthermore the definition of the c_{ij}'s following from the virtual load technique (see Eq. 7) clearly indicates that

$$u_j^o = \sum_{i=1}^{n} c_{ij}^o x_i^o \tag{12}$$

Therefore (10) can be rewritten

$$\tilde{h}_j(x) = \bar{u}_j - \sum_{i=1}^{n} c_{ij}^o \, x_i \geq 0 \tag{13}$$

which is equivalent to (8) when recast in terms of the direct variables a_i. It is thus apparent that a unified approach to structural weight minimization of finite element systems has emerged, which consists in replacing the initial problem (1-3) with a sequence of explicit approximate - or linearized - problems of the following form :

$$\text{minimize} \quad W = \sum_{i=1}^{n} \rho_i \, a_i \tag{14}$$

$$\text{subject to} \quad \bar{u}_j - \sum_{i=1}^{n} \frac{c_{ij}}{a_i} \geq 0 \tag{15}$$

$$\bar{a}_i \geq a_i \geq \underline{a}_i \tag{16}$$

The GOC statement results from writing the KUHN-TUCKER optimality conditions for the problem (14-16). This yields an explicit expression for the design variables a_i in terms of the lagrangian multipliers r_j associated with the behavior constraints (15) (see Ref [6, 8] for more details) :

$$a_i = \left(\frac{1}{\ell_i} \sum_{j=1}^{m} c_{ij}\, r_j\right)^{1/2} \quad \text{if} \quad \ell_i \underline{a}_i^2 < \sum_{j=1}^{m} c_{ij}\, r_j < \ell_i \bar{a}_i^2 \qquad (17)$$

$$a_i = \underline{a}_i \quad \text{if} \quad \sum_{j=1}^{m} c_{ij}\, r_j \leq \ell_i \underline{a}_i^2 \qquad (18)$$

$$a_i = \bar{a}_i \quad \text{if} \quad \sum_{j=1}^{m} c_{ij}\, r_j \geq \ell_i \bar{a}_i^2 \qquad (19)$$

Of course the lagrangian multipliers must be nonnegative, more precisely, they must satisfy the complementary conditions :

$$r_j > 0 \quad \text{if} \quad \sum_{i=1}^{n} \frac{c_{ij}}{a_i} = \bar{u}_j \qquad (20)$$

$$r_j = 0 \quad \text{if} \quad \sum_{i=1}^{n} \frac{c_{ij}}{a_i} < \bar{u}_j \qquad (21)$$

In order to compute the lagrangian multipliers satisfying (20-21), an interesting approach is to resort to dual methods, which leads to maximize the lagrangian function considered as a function of the lagrangian multipliers - or dual variables - only (see Refs [1, 6, 8] and section 5). Once the solution of the dual problem has been found, the corresponding optimal design variables are easily computed from the explicit optimality criteria relations (17-19). Note that the design variables can be separated into a group of active (or free) variables (see Eq. 17) and a group of passive (or fixed) variables (see Eqs. 18 and 19). This subdivision into active and passive design variable groups is classical in the optimality criteria approaches [3, 9]. It corresponds to the fact that the dual space - i.e. the space of the lagrangian multipliers r_j - is partitioned into several regions separated by planes across which the second derivatives of the dual function are discontinuous [6, 8].

The whole process of combining the linearization of the behavior constraints with respect to the reciprocal design variables and a dual solution scheme can be viewed as a generalization of the optimality criteria approaches. It is important to mention that this basic approach of converting the initial problem into a sequence of explicit subproblems is now widely recognized [6, 10] and it is routinely employed for large scale industrial applications [11].

3. PURE BENDING ELEMENTS

In this section attention is focused on discretized models made up of pure beam and plate elements subjected to flexural loads only (for more details, see section 6 of Ref. [2]). The stiffness matrix of such a bending element is usually not merely proportional to its cross-sectional size and therefore the optimization strategy reviewed in the previous section must be modified. The way to deal with a beam element subjected to uniaxial bending depends upon the relationship between the principal moment of inertia I and the cross-sectional area a. A wide variety of situations is taken into consideration by adopting the following relation :

$$I = c\ a^p \qquad (22)$$

where c is a constant that depends only on the shape of the beam cross-section and p is a positive number.

Most of the time p is taken as an integer number, equal to 1, 2 or 3. The case p = 1 corresponds to thin walled beams, for example, sandwich beams, pipes with fixed diameter and variable thickness, etc... The GOC approach of section 2 remains then fully applicable, since the stiffness matrix continues to be linear in the design variables. The case p = 2 is that of beams with uniformly varying cross-section. The shape of the cross-section is kept unvariant while its area is modified during redesign (dilatation or contraction). Finally the case p = 3 is concerned with beams having full cross-section whose height varies while other sizes are fixed. For a beam subjected to pure bending, the flexural rigidity is proportional to the moment of inertia and therefore, in a finite element context, the structural stiffness matrix exhibits the following explicit form in terms of the cross-sectional areas :

$$K = \sum_{i=1}^{n} K_i = \sum_{i=1}^{n} a_i^p\ \bar{K}_i \quad , \quad p > 0 \qquad (23)$$

where each matrix $\bar{K}_i$ is independent of the design variables a_i.

With regard to plate elements subjected to pure bending, two cases must be distinguished. The first case is that of sandwich plates with constant core thickness. The sheet thicknesses constitute then the design variables. Consequently the stiffness matrix continues to depend linearly on the design variables and the GOC relations (17-21) remain fully applicable. The second case is concerned with full plates with variable thickness. The stiffness is then proportional to the cube of the thickness and, in an assembling of plate elements, the relation (22) must be chosen with $p = 3$.

By assuming that the structural discretization is made up entirely of elements of the same type, the stiffness matrix exhibits the form (23), where p takes on the same value for each member. In these circumstances, the GOC can be derived just as in the case of thin walled structures, by adopting a change of variables tending to reduce the nonlinear character of the constraints :

$$x_i = \frac{1}{a_i^p} \tag{24}$$

The next step is to linearize the constraints with respect to the new variables x_i, which requires gradient evaluation (see Eq. 10). Restricting again the discussion to stress and displacement constraints, it is easily shown (see Ref. [2]) that the first order Taylor series expansion (10) reduces to the form (13), or, when written in terms of the direct variables :

$$\tilde{h}_j(a) \equiv \bar{u}_j - \sum_{i=1}^{n} \frac{c_{ij}}{a_i^p} \geq 0 \tag{25}$$

The c_{ij} coefficients can be interpreted as the gradients of the response quantities with respect to the intermediate variables x_i defined in (24), but they can also be related to the virtual strain energies e_{ij} in the structural members :

$$c_{ij} = q^T \bar{K}_i \, q_j \, a_i^{2p} = q^T K_i \, q_j \, a_i^p = e_{ij} \, a_i^p \tag{26}$$

where, by definition,

$$e_{ij} = q^T K_i q_j \tag{27}$$

In this connection it should be recognized that virtual load considerations could directly be employed to derive the explicit approximations (25), instead of resorting to first order Taylor series (see next section).

The optimality criteria equations are very similar to (17-19), provided that care is taken of the exponent p appearing in (25). The basic redesign relations for the active design variables must read as follows :

$$a_i = \left(\frac{p}{\ell_i} \sum_{j=1}^{m} c_{ij} r_j\right)^{\frac{1}{p+1}} \tag{28}$$

where it is understood that the dual variables r_j (i.e. the lagrangian multipliers) must satisfy the complementarity conditions (see Eqs. 20 and 21). A physical interpretation of the optimality criterion is obtained by introducing the virtual strain energy densities per unit weight :

$$\varepsilon_{ij} = \frac{e_{ij}}{\ell_i a_i} \tag{29}$$

where e_{ij} is defined in (27). In terms of the ε_{ij}'s, the optimality criterion (28) takes the "energetic" form

$$\sum_{j=1}^{m} r_j \varepsilon_{ij} = \text{constant} \tag{30}$$

In the special case where only one displacement constraint is specified, the optimality criterion states that the virtual strain energy density must be the same in each element. In this simple case, it is possible to solve analytically the explicit problem and to derive explicit redesign relations in terms of known quantities. The active design variables can be shown to be given by

$$a_i = \left[\frac{1}{\bar{u} - u_o} \sum_k \left(\ell_k^{\frac{p}{p+1}} c_k^{\frac{1}{p+1}}\right)\right]^{\frac{1}{p}} \left(\frac{c_i}{\ell_i}\right)^{\frac{1}{p+1}} \tag{31}$$

while the remaining passive variables are fixed to an upper or a lower limit (u_o denotes the contribution of these passive variables to the displacement constraint $u \leq \bar{u}$). It is worth mentioning that (31) is well suited for the design of plates in bending with a single displacement constraint. Since then $p = 3$ the redesign relation (31) involves the fourth root of the coefficients c_i, rather than the third root as employed in Ref. [12] on an intuitive basis. Note also that by taking $p = 1$ in (31), conventional redesign relations are recovered, which were devised for trusses [3], sandwich beams [13], etc...

It can be concluded that the generalized optimality criteria approach can easily be extended to deal with pure bending elements by defining adequate intermediate variables. High quality explicit approximations of the behavior constraints can still be generated and the resulting GOC keeps its interpretation in terms of energy densities in the structural members. Seeking the design variables that satisfy the GOC at each redesign stage can still be achieved efficiently by resorting to dual methods, because the explicit approximate problem remains separable and strictly convex when expressed in the intermediate design variables x_i.

4. FLEXION-EXTENSION ELEMENTS

When flexion and extension loadings act simultaneously with comparable intensity at the element level, the definition (23) of the stiffness matrix can no longer characterize the structural model with sufficient accuracy. To help fix ideas, consider a flat shell element made up of a membrane and a plate stacked together. The stiffness matrix of such a flat shell element exhibits the form

$$K_i = a_i K_i^{(1)} + a_i^3 K_i^{(3)} \tag{32}$$

where $K_i^{(1)}$ and $K_i^{(3)}$ are constant matrices. As a result, in the GOC approach, if the constraints are linearized with respect to the reciprocal design variables (9), their first order explicit approximations, given by expressions similar to (8), will be of high quality only if the structural members behave mainly in extension. On the other hand, if the bending behavior is dominant, it is better to adopt the change of variables (24), yielding first order explicit approximations of the form (25) (with p = 3). As a matter of fact, the true situation is usually a combination of extension and bending. In a practical structure, some members work mainly in extension, some in flexion, and others, simultaneously in flexion and extension. Whence the idea of using the following explicit approximations, which should be valid in any situation :

$$\tilde{h}_j(a) \equiv \bar{u}_j - \sum_{i=1}^{n} \left(\frac{c_{ij}^{(1)}}{a_i} + \frac{c_{ij}^{(3)}}{a_i^3}\right) \geqslant 0 \tag{33}$$

where the coefficients $c_{ij}^{(1)}$ and $c_{ij}^{(3)}$ are considered as constant throughout the redesign phase.

Because it is no longer possible to select appropriate intermediate design variables, the explicit approximations (33) cannot be obtained by merely using first order Taylor series as in the case of pure bending elements. However an essential requirement is that these approximations remain correct up to

the first order, despite the fact that they do not result from a strict linearization process. In other words the following equality must hold :

$$\frac{\partial \tilde{h}_j}{\partial a_i}(a^o) = \frac{\partial h_j}{\partial a_i}(a^o) = \frac{c_{ij}^{(1)}}{(a_i^o)^2} + 3 \frac{c_{ij}^{(3)}}{(a_i^o)^4} \tag{34}$$

This condition insures that, at the optimum, the solution to the explicit approximate problem satisfies the (first order) optimality conditions of the real problem, that is, the approximate and real restraint surfaces have the same tangent plane (see Fig. 1). As a result, the GOC approach should converge to a true (at least local) minimum weight design. It will be shown in this section how such first order explicit approximations can be obtained for various types of behavior constraints and structural models.

4.1. Displacement Constraints

The key idea to obtain explicit approximations of the displacement constraints is to come back to the virtual load procedure, which permits decomposing any static response quantity into the contributions of each element. The expression (6) can be rewritten in the more general form

$$u_j = q^T K q_j = \sum_{i=1}^{n} q^T K_i q_j \tag{35}$$

On the other hand, it can be proved that the gradient of u_j is given by (see Ref. [1], p. 74) :

$$\frac{\partial u_j}{\partial a_i} = - q^T \frac{\partial K}{\partial a_i} q_j = - q^T \frac{\partial K_i}{\partial a_i} q_j \tag{36}$$

Now, for a rather general class of structural models, each element stiffness matrix can be assumed to have the following explicit form in terms of its design variable [14] :

$$K_i = \sum_{p=1}^{3} a_i^p K_i^{(p)} \tag{37}$$

where the matrices $K_i^{(p)}$ are independent of the design variables. Note that most often, at least one of the $K_i^{(p)}$, p = 1, 2, 3, is zero (for example $K_i^{(2)}$ is zero in the stiffness matrix (32) of a flat shell element). Introducing (37) into (35), it appears that a convenient explicit approximation of a displacement constraint (5) is :

$$\tilde{h}_j(a) = \bar{u}_j - \sum_{i=1}^{n} \sum_{p=1}^{3} \frac{c_{ij}^{(p)}}{a_i^p} \geq 0 \tag{38}$$

where the coefficients

$$c_{ij}^{(p)} = (q^T K_i^{(p)} q_j)\, a_i^{2p} \tag{39}$$

are assumed to be constant during the current stage. The gradient of this explicit approximate constraint is

$$\frac{\partial \tilde{h}_j}{\partial a_i} = \sum_{p=1}^{3} \frac{p\, c_{ij}^{(p)}}{a_i^{p+1}} \tag{40}$$

On the other hand, differentiating (37) and inserting the result into (36) shows that

$$\frac{\partial u_j}{\partial a_i} = - \sum_{p=1}^{3} p\, a_i^{p-1}\, q^T K_i^{(p)} q_j \tag{41}$$

Therefore it can be concluded that the expressions (38) represent first order explicit approximations, in that they restitute the exact values of the constraints and their first partial derivatives at the design point a^o where the structural analysis is made :

$$\left\{ \begin{array}{ll} \tilde{h}_j(a^o) = h_j(a^o) = \bar{u}_j - \sum_{i=1}^{n} \sum_{p=1}^{3} \frac{c_{ij}^{(p)}}{(a_i^o)^p} & (42) \\ \frac{\partial \tilde{h}_j}{\partial a_i}(a^o) = \frac{\partial h_j}{\partial a_i}(a^o) = \sum_{p=1}^{3} \frac{p\, c_{ij}^{(p)}}{(a_i^o)^{p+1}} & (43) \end{array} \right.$$

An alternative approach is to employ simple linear approximations in the reciprocal design variables (9), i.e. the first order Taylor series expansions (10). Then the explicit subproblem exhibits exactly the same form (14-16) as in the case of thin walled structures and dual methods need not be modified as they will be in section 5. Of course, as previously mentioned, the convergence of the overall optimization process might be lowered, or even become unstable, in the case where the bending behavior is dominant in most of the elements. The reader is referred to sections 6.1 and 6.5 for numerical examples comparing the explicit approximations (8), (25) and (38).

4.2. Stress Constraints

The situation is much more delicate for stress constraints than for displacement constraints. In contrast to the case of thin walled structures modeled by bar and membrane elements, the stress matrices are no longer constant in the case of flexion-extension elements. For illustration consider again a flat shell element. The stresses in the upper sheet can be computed in terms of the generalized displacements by

$$\sigma_k = T_k q = (T_k^{(m)} + a_k\, T_k^{(b)})q = \sigma_k^{(m)} + \sigma_k^{(b)} \qquad (44)$$

where σ_k, $\sigma_k^{(m)}$, $\sigma_k^{(b)}$ are matrix representations of the total stress, membrane stress and bending stress tensors in member k, and T_k, $T_k^{(m)}$, $T_k^{(b)}$ are the corresponding stress matrices, which are independent of the design variables. Each displacement component admits a first order explicit approximation of the form (see Eq. 33)

$$\tilde{q}_j = \sum_{i=1}^{n} \left(\frac{c_{ij}}{a_i} + \frac{d_{ij}}{a_i^3}\right)$$

Therefore, a natural choice for the approximation of a stress component could be as follows :

$$\tilde{\sigma}_k = \sum_{i=1}^{n} \left(\frac{c_{ij}^{(m)}}{a_i} + \frac{d_{ij}^{(m)}}{a_i^3}\right) + a_k \sum_{i=1}^{n} \left(\frac{c_{ij}^{(b)}}{a_i} + \frac{d_{ij}^{(b)}}{a_i^3}\right) \qquad (45)$$

The terms containing $c_{ij}^{(m)}$ and $d_{ij}^{(m)}$ arise from the membrane part of the stress $\sigma_k^{(m)}$, and those containing $c_{ij}^{(b)}$ and $d_{ij}^{(b)}$, from the bending stress $\sigma_k^{(b)}$.

The expression (45) is still a first order approximation. It is no longer separable, but keeps a simple algebraic form suitable to specialized algorithms such as those based on dual methods. However, when considering the true stress limitation (e.g. upper limit on the Von Mises stress), the explicit constraint becomes much more complicate. For these reasons, and before finding something better, it has been decided to follow the alternative strategy proposed at the end of section 4.1, that is, to employ simple linear approximations of the form (10). To compute the gradient of the stress constraint with respect to the reciprocal variables (9), we note that, from the equivalent Von Mises stress

$$\sigma_{ek} = (\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3 \tau_{xy}^2)_k^{1/2} \tag{46}$$

it comes

$$\frac{\partial \sigma_{ek}}{\partial x_i} = \frac{1}{2x_i}\left[(2 \sigma_x - \sigma_y)\frac{\partial \sigma_x}{\partial x_i} + (2 \sigma_y - \sigma_x)\frac{\partial \sigma_y}{\partial x_i} + 6 \tau_{xy} \frac{\partial \tau_{xy}}{\partial x_i}\right]_k \tag{47}$$

On the other hand, from (44) it follows that the derivative of any stress component σ_x, σ_y or τ_{xy} has the form

$$\frac{\partial \sigma_k}{\partial x_i} = T_k \frac{\partial q}{\partial x_i} - \delta_{ik} \frac{\sigma_k^{(f)}}{x_i^2} \tag{48}$$

where δ_{ik} is the Kronecker symbol.

4.3. Frequency Constraints

Constraints on natural frequencies usually consist in imposing lower limits

$$h_j(a) \quad \omega_j^2(a) - \underline{\omega}_j^2 \geqslant 0 \qquad j = 1,m \tag{49}$$

They are directly written in terms of the squares of the frequencies, because these quantities naturally appear in the eigenproblem characterizing the structural modal analysis :

$$K\, q_j - \omega_j^2 M q_j = 0 \qquad (50)$$

In this equation K and M represent the stiffness and mass matrices, and (q_j, j = 1,m) are the modal displacements, i.e., the eigenvectors solution of (50), associated with eigenvalues ω_j^2. *The structural mass matrix has a linear form in terms of the* design variables :

$$M = \sum_{i=1}^{n} M_i + M_c = \sum_{i=1}^{n} a_i\, \bar{M}_i + M_c \qquad (51)$$

where $\bar{M}_i$ and M_c are independent of the design variables. $\bar{M}_i$ denotes the mass matrix of the ith element when $a_i = 1$. M_c represents the contribution of the non-structural masses, such as equipments, fuel, etc... It is well known that the first derivatives of the frequencies with respect to the design variables are given by [see for example Ref. [2], section 4.2] :

$$\frac{\partial \omega_j^2}{\partial a_i} = \frac{1}{m_j}\, q_j^T \left(\frac{\partial K_i}{\partial a_i} - \omega_j^2 \frac{\partial M_i}{\partial a_i}\right) q_j \qquad (52)$$

where m_j is the generalized mass of the jth mode :

$$m_j = q_j^T\, M\, q_j \qquad (53)$$

The way to derive first order explicit approximations of the frequency constraints is less apparent than for displacement constraints. In this report, guided by the work done in Ref [2] (section 4.2), *the following decomposition* of the eigenvalues in terms of the stiffness and mass contributions of each element will be used :

$$\omega_j^2 = \omega_j^2 \left(1 + \frac{\bar{m}_j}{m_j}\right) - \frac{1}{m_j} \sum_{i=1}^{n} q_j^T (K_i - \omega_j^2\, M_i) q_j \qquad (54)$$

where

$$\bar{m}_j = q_j^T\, M_c\, q_j = m_j - \sum_{i=1}^{n} q_j^T\, M_i\, q_j \qquad (55)$$

represents the contribution of the non structural masses to the generalized mass m_j (see Eq. 53). By taking account of the explicit definitions (37) of the stiffness matrices K_i and (51) of the mass matrices M_i, the high quality explicit approximations of the frequency constraints take the form (38), with

$$\bar{u}_j = \omega_j^2 \left(1 + \frac{\bar{m}_j}{m_j}\right) - \underline{\omega}_j^2 \qquad (56)$$

$$c_{ij}^{(p)} = \frac{q_j^T \left(K_i^{(p)} - \omega_j^2 \bar{M}_i \delta_{1p}\right) q_j}{m_j} a_i^{2p} \qquad p = 1,2,3 \quad (57)$$

where $\delta_{1p} = 1$ only if $p = 1$ and is 0 otherwise.

The coefficients $c_{ij}^{(p)}$ and the modified limits $\bar{u}_j$ are frozen to their values at the current design point. Just as for the displacement constraints, it is easily verified that the expressions (38) remain first order explicit approximations satisfying (42, 43). As a matter of fact, they can be interpreted as first order Taylor series expansions in terms of $1/x_i$, $1/x_i^2$, $1/x_i^3$ considered as independent variables.

4.4. Buckling Constraints

Just as the natural frequencies, the critical load factors λ_j are defined through an eigenproblem

$$K\, q_j - \lambda_j\, S\, q_j = 0 \qquad (58)$$

where S represents the geometric stiffness matrix and (q_j, $j = 1,m$) denote the eigenvectors solution of problem (58), associated with eigenvalues λ_j. The physical meaning of the q_j's is that of displacements in the jth buckling mode , for a critical load factor λ_j. The buckling constraints consist in imposing lower limits on the buckling loads :

$$\lambda_j \geq \underline{\lambda}_j \qquad j = 1, m \qquad (59)$$

In this work, the following form of the constraints will be adopted :

$$h_j(a) = \frac{1}{\underline{\lambda}_j} - \frac{1}{\lambda_j(a)} \geq 0 \qquad j = 1, m \tag{60}$$

because it has been found that better explicit approximations are generated when expanding the reciprocal of the buckling loads rather then the λ_j's themselves.

The stiffness matrix K has the form (23). The geometric stiffness matrix is related to the initial stress state in the elements and therefore it depends implicitly on all the design variables :

$$S = \sum_{i=1}^{n} S_i(a) \tag{61}$$

It is worth recalling that the matrices S_i are independent of the design variables for a statically determinate structure.

As explained in section 5.2 of Ref [2], the first derivatives of the buckling loads are given by

$$\frac{\partial \lambda_j}{\partial a_i} = \frac{1}{q_j^T\ S\ q_j}\ q_j^T \left(\frac{\partial K_i}{\partial a_i} - \lambda_j\ \frac{\partial S}{\partial a_i}\right) q_j \tag{62}$$

In opposition with the static and dynamic cases previously discussed, the derivatives appearing in (62) are not directly available, because the elements of the geometrical stiffness matrix are function of the stresses acting in the prebukling state. However, by assuming that the terms $\frac{\partial S}{\partial a_i}$ are negligible, the gradients (62) become easily computable. This assumption, which is typical of optimality criteria approaches for static constraints, amounts to not taking into account the effects of structural redundancy :

$$\frac{\partial S}{\partial a_i} = 0 \qquad i = 1, n \tag{63}$$

In this report, the following decomposition of the reciprocal buckling loads in terms of the contribution of each element will be used :

$$\frac{1}{\lambda_j} = \frac{1}{s_j \lambda_j} \sum_{i=1}^{n} q_j^T K_i q_j \tag{64}$$

where

$$s_j = q_j^T S q_j \tag{65}$$

Substituting the explicit definition (37) of the stiffness matrices K_i into (64) yields the high quality explicit approximations of the buckling load constraints (60) under the form (38), with

$$\bar{u}_j = \frac{1}{\underline{\lambda}_j} \tag{66}$$

$$c_{ij}^{(p)} = \frac{q_j^T K_i^{(p)} q_j}{s_j \lambda_j^2} a_i^{2p} \qquad p = 1, 2, 3 \tag{67}$$

Again, the coefficients $c_{ij}^{(p)}$ and the modified limits $\bar{u}_j$ are frozen to their values at the current design point. Just as for the displacement and frequency constraints, it is easily verified that the expressions (38) are first order explicit approximations of the constraints (60), satisfying (42,43), provided that the assumption (63) can be introduced into the gradient (62) (statical determinacy).

The reason why the buckling constraints are treated by expanding the reciprocal critical loads (see Eq. 64) is that, for a thin walled structure, the approximation (38) takes the form (8), where the coefficients c_{ij} remain constant along the scaling line. This property, which is well detailed in section 3 of Ref [2], has been found very important for the case of stress and displacement constraints. Geometrically it means that the real restraint surface $h_j(a) = 0$ is replaced by a tangent surface $\tilde{h}_j(a) = 0$ at its point of intersection with the scaling line (see Fig. 3 of Ref. [2]).

5. SOLUTION OF THE EXPLICIT PROBLEM

From the foregoing developments, it appears that, at each stage of the optimization process, the following subproblem must be solved

$$\text{minimize} \qquad W = \sum_{i=1}^{n} \ell_i \; a_i \tag{68}$$

$$\text{subject to} \qquad \sum_{i=1}^{n} \sum_{p=1}^{3} \frac{c_{ij}^{(p)}}{a_i^p} \leq \bar{u}_j \tag{69}$$

$$\underline{a}_i \leq a_i \leq \bar{a}_i \tag{70}$$

Unless the summation on p in (69) is restricted to a single value of p (i.e. approximation of the form (25) with p = 1, 2 or 3), it is no longer possible to find intermediate variables [i.e. x_i given in (24)] in terms of which the explicit constraints (69) would be linear. Therefore, the primal solution of problem (68-70) is more difficult to achieve if a gradient projection type of algorithm is employed as in the mixed method developed in a previous work for thin walled structures [1, 15]. The expressions (69) are still explicit and they continue to exhibit a simple algebraic form. Consequently a general purpose optimization algorithm such as NEWSUMT [16] could easily be adapted to take the constraints (69) into account. However, because they are still additively separable, resorting to dual methods remains probably the best strategy, just as in the case of thin walled structures [1, 2, 6, 8].

The minimization problem (68-70) can be solved efficiently as an auxiliary maximization problem in the m lagrangian multipliers r_j associated with the explicit behavior constraints (69). This dual problem reads as follow [8] :

$$\text{maximize} \qquad \ell(r) = \sum_{i=1}^{n} \ell_i \; a_i(r) + \sum_{j=1}^{m} r_j \; g_j(r) \tag{71}$$

$$\text{subject to} \qquad r_j \geq 0 \qquad\qquad j = 1, m \tag{72}$$

where $g_j(r)$ denote the components of the dual function gradient, which are equal to the values of the primal constraints :

$$g_j(r) = \frac{\partial \ell}{\partial r_j}(r) = \sum_{i=1}^{n} \sum_{p=1}^{3} \frac{c_{ij}^{(p)}}{a_i^p} - \bar{u}_j \qquad (73)$$

The primal variables $a_i(r)$ are related to the dual variables r_j through the following quasi-unconstrained minimization problem (see for example Ref [6]) :

$$\text{minimize} \quad \sum_{i=1}^{n} \ell_i a_i + \sum_{j=1}^{m} r_j \left(\sum_{i=1}^{n} \sum_{p=1}^{3} \frac{c_{ij}^{(p)}}{a_i^p} - \bar{u}_j \right) \qquad (74)$$

$$\text{subject to} \quad \underline{a}_i \leq a_i \leq \bar{a}_i \qquad (75)$$

Because of the separability of this problem, it can be decomposed into n one-dimensional minimization problems of the form

$$\min_{\underline{a}_i \leq a_i \leq \bar{a}_i} \left[\ell_i a_i + \sum_{p=1}^{3} \frac{C_i^{(p)}}{p\, a_i^p} \right] \qquad (76)$$

where

$$C_i^{(p)} = p \sum_{j=1}^{m} r_j c_{ij}^{(p)} \qquad (77)$$

Setting to zero the first derivative of the single variable function appearing in (76), it is seen that the $\check{n}$ "active" design variables can be obtained by solving the nonlinear algebraic equations

$$\sum_{p=1}^{3} \frac{C_i^{(p)}}{a_i^{p+1}} = \ell_i \qquad i = 1, n \qquad (78)$$

Note however that when solving (78), the side constraint (75) must be taken into account and that, if several values satisfy equation (78), the one that minimizes the function (76) must be retained. Standard techniques can be employed to deal with such a simple one-dimensional problem (see e.g. [17]).

In many cases, as previously mentioned, at least one of the terms in the summation on p in (76) and (78) disappears, and it is possible to treat the problem analytically. As an illustration, consider again the case of a flat shell element, where the term in p = 2 is missing. Problem (76) exhibits then the form (omitting the index i) :

$$\min_{a \geq \bar{a}} \left\{ \ell a + \frac{C^{(1)}}{a} + \frac{C^{(3)}}{3a^{3}} \right\} \tag{79}$$

while equation (76) reads as follow

$$\ell - \frac{C^{(1)}}{a^{2}} - \frac{C^{(3)}}{a^{4}} = 0 \tag{80}$$

or, setting

$$z = \frac{1}{a^{2}} \tag{81}$$

$$\ell - C^{(1)} z - C^{(2)} z^{2} = 0 \tag{82}$$

The constant ℓ is known to be positive, and everything depends thus on the sign of the constants $C^{(1)}$ and $C^{(3)}$, which represent the contribution of the membrane and the bending, respectively. In general, equation (82) admits two solutions, and the one that renders minimal the function (79) must be adopted.

From the foregoing developments, it appears that the dual function (71) can be considered as a function of the dual variables only. The dual problem stated in (71, 72) exhibits an attractive feature, namely, it is a quasi-unconstrained problem, because taking care of the nonnegativity constraints (72) on the dual variables is straightforward. Given some nonnegative dual variables, the corresponding primal variables are computed by solving (76) or (78) and the primal constraints are evaluated by using (73). The dual function (71) and its gradient (73) are then directly known and a feasible ascent direction can therefore be determined. In a second order algorithm, the hessian matrix of the dual function has to be computed :

$$H_{jk} = \frac{\partial^{2} \ell}{\partial r_{j} \partial r_{k}} = \frac{\partial g_{j}}{\partial r_{k}} \tag{83}$$

From (73), it comes

$$\frac{\partial g_{j}}{\partial r_{k}} = - \sum_{i=1}^{n} \sum_{p=1}^{3} \frac{p\, c_{ij}^{(p)}}{a_{i}^{p+1}} \frac{\partial a_{i}}{\partial r_{k}} \tag{84}$$

and by differentiating the definition (78) of $a_i(r)$ with respect to r_k, it follows that .

$$-\left(\sum_{p=1}^{3} c_i^{(p)} \frac{(p+1)}{a_i^{p+2}}\right) \frac{\partial a_i}{\partial r_k} + \sum_{p=1}^{3} \frac{\partial c_i^{(p)}}{\partial r_k} \frac{1}{a_i^{p+1}} = 0 \qquad (85)$$

On the other hand, it is easily seen from (77) that

$$\frac{\partial c_i^{(p)}}{\partial r_k} = p \; c_{ik}^{(p)} \qquad (86)$$

Regrouping terms, it can finally be concluded that the terms of the hessian matrix are given by

$$H_{jk} = - \sum_{i=1}^{n} \frac{\left(\sum_{p=1}^{3} p \; c_{ij}^{(p)} \; a_i^{-1-p}\right)\left(\sum_{p=1}^{3} p \; c_{ik}^{(p)} \; a_i^{-1-p}\right)}{\sum_{p=1}^{3} (p+1) \; c_i^{(p)} \; a_i^{-2-p}} \qquad (87)$$

where the summation on i is restricted to the n active primal variables, that is, those that are not fixed to a lower or an upper limit (of course $\frac{\partial a_i}{\partial r_k} = 0$ for a passive variable). Knowing the gradient (73) and the hessian matrix (87) furnishes the Newton search direction

$$z = - H^{-1} \; g \qquad (88)$$

The next dual point is then given by

$$r^{+} = r + \alpha z \qquad (89)$$

where α is the step length taken along the direction z. Most often a regular Newton method unit step [$\alpha = 1$ in (89)] is selected, however, the value of α must sometimes be lowered to prevent one of the dual variables from becoming negative [6].

The second order dual optimizer implemented in SAMCEF [18] has been especially devised so that it seeks the maximum of the dual function by operating in a sequence of dual subspaces with gradually increasing dimension. In this way, the effective dimensionality of the maximization problem never exceeds the number of active behavior constraints, which correspond to non-zero dual variables. Past experience with thin walled structures indicate that this number is relatively small in practice, which explains the remarkable efficiency of the dual method [6] or

generalized optimality criterion [1, 7] approach.

The basic second order dual algorithm, first introduced in [1, 8] employed an exact line search to find the step length t^* that maximizes the dual function along the search direction. Subsequently the algorithm was modified by introducing a simplified line search procedure where a unit step length is adopted most of the time. A detailed description of this algorithm can be found in Ref [6]. The latest version, including the treatment of explicit constraints of the form (69) is very similar. The only differences are the way of computing the primal variables in terms of the dual variables [see Eqs (76,78)] and the formula employed for the hessian matrix [see Eq (87)]. It should however be recognized that numerical difficulties might occur due to the lack of convexity of the explicit constraints (69). Indeed it is possible that the solution of Eq (76) for a primal variable suddenly jumps from a fixed lower or upper bound to a free value when the dual variables are slightly modified. This phenomenon, which means that the dual function first derivatives are discontinuous along some surfaces, needs more investigation.

6. NUMERICAL APPLICATIONS

All the examples presented involve rather sophisticated flat shell elements that are characterized by a displacement field cubic in extension and quintic in flexion (hybrid quadrangular flat shell) [18].

6.1. Cantilever Beam with End Moment

The first example is concerned with a cantilever beam loaded with a concentrated moment at its free end [see Fig. 2]. This problem has been previously solved numerically by PRASAD and HAFTKA [19] using an extended interior penalty function formulation for a beam having the following properties : length = 10 in, width = 1 in, applied moment = 540 in.lb , Young's modulus = 10^7 psi, Poisson's ratio = 0.3 and mass density = 0.3 lb/in^3. There exists also an analytical solution, which was obtained by HAUG [20]. The analytical optimum design for a displacement limit of 0.5 in at the free end and specified allowable stresses of 30,000 psi is as follows (see Fig. 2) :

$$a(x) = 0.30 \qquad \text{for} \qquad 0 < x < 2.3 \tag{90}$$

$$a(x) = 0.244\ x^{1/4} \qquad \text{for} \quad 2.3 < x < 10 \tag{91}$$

It is worth noticing that the beam being statically determinate, the redesign relation (31) is exact (with p = 3) and reduces to (90, 91). Therefore the optimal solution should be generated after one structural analysis only.

The structure is discretized using 20 quadrilateral plate bending finite elements as indicated in Fig. 2. There is no design variable linking and the problem involves thus 20 independent design variables. The problem was first solved by using the cubic expansions (25) with p = 3 and the dual algorithm described in section 5. As previously stated, because the structure is statically determinate, the explicit problem (68-70) is exact and the optimum design is obtained in one single analysis whatever may be the initial thickness a^o. Then, the same example

was again solved by using the linear expansion (8) with an initial thickness $a^o = 0.5$ in, resulting in an approximate explicit problem of the form (14-16). The iteration history is given in Table 1. It can be seen that, surprisingly enough, the use of simple linear expansions gives rise to satisfactory results.

The two final designs obtained are listed in Table 2, together with the analytical solution corresponding to Eqs. (90, 91) and the numerical results of Ref [19]. A close agreement between all the final optimal designs can be observed.

6.2. Simply Supported Square Plate with Deflection Constraint

The second example consists in minimizing the weight of the simply supported square plate shown on Fig. 3. It is subjected to a concentrated load of 1000 kN at its center, where the deflection is limited to 0.02 m. By symmetry only a quarter of the plate has to be analyzed. The mesh involves 25 plate elements and 175 degrees of freedom. The material properties are as follows : Young's modulus $E = 2.10^{10}$ N/m^2, Poisson's ratio $\nu = 0.3$ and weight density $\rho = 7800$ kg/m^3. The dimensions of the plate are 10 m by 10 m. The initial thickness is $a^o = 0.05$ m and the minimum thickness is 0.005 m. Because the problem involves only one behavior constraint, the redesign relations (31) can be applied with p = 1. However, the structure being statically indeterminate, they must be employed recursively. The iteration history data are given in Table 3 and the final design is illustrated in Fig. 5. It is worth pointing out that only 10 reanalyses are sufficient to generate a nearly converged solution, while in a similar problem solved in Ref [12] by using p = 2 in the redesign relations (31), almost 50 iterations were required.

6.3. Simply Supported Square Plate with Frequency Constraint

Attention is now focused on the simply supported square plate shown in Fig. 4. This example is taken from Ref [21] and it is

concerned with the minimum weight design of the plate subject to a natural frequency constraint. The dimensions of the plate are 10 in by 10 in and its material properties are Young's modulus = 30 x 10^6 lb/in^2 and Poisson's ratio = 0.3. In Ref [21] it is stated that the uniform plate with $\rho\underline{\omega}^2$ = 1400 is taken as the initial estimate to the optimization problem, where ρ is the mass density and $\underline{\omega}$, the minimum fundamental frequency. Therefore, in the present study, assuming steel material with mass density ρ = 0.283 lb/in^3, the minimum frequency was chosen as $\underline{\nu} = \underline{\omega}/2\pi$ = 11.2 Hz. Iteration history data are presented in Table 4 for three cases differing by the minimum thickness constraint ($\underline{a}$ = 0.1, 0.05 and 0.001 in). The initial design in cases 1 and 3 corresponds to a uniform thickness a^o = 0.2 in, while in case 2, a^o equals 0.12 in. Final designs are illustrated in Fig. 4. By symmetry only a quadrant of the plate has to be analyzed and designed. It can be observed that the design obtained in Ref [21] is different from the design achieved in this study (case 1). However both designs have about the same weight (0.765 lb and 0.752 lb). It is also worth pointing out that the design of plates in bending is an analytically difficult problem, leading to multiple local optima and suggesting that the optimal plate should be made up of an infinite number of stiffeners [22].

6.4. I-beam Structure

The third example involves the I-beam structure schematized in Fig. 5. The problem consists in minimizing the weight of the beam while imposing lower bounds on the frequencies of the three first eigenmodes : flange flexion, torsion and web flexion. Detailed data can be found in Ref. [2]. In a first optimization exercise, a pure membrane model was employed. It involves 35 second degree displacement elements, including 10 fictitious diaphragms (without masses). These dummy members are introduced to obtain a satisfactory representation of the torsional mode. Only 5 analyses are sufficient to generate an optimum design for this membrane model.

However, when this final design was analyzed by using a more accurate model made up of flat shell elements, the torsional frequency (mode 2) was seen to be violated by 10 %. Therefore the problem was again solved with this new model, by resorting to the theory proposed in section 4.3 and to the dual optimizer described in section 5. Iteration history data are illustrated in Fig. 5 and the final designs are given in Table 5 for both finite element models of the I-beam. It can be seen that the use of flat shell elements, although yielding slower convergence, gives satisfactory results.

6.5. U-beam Structure

In an attempt to consider a case where both flexion and extension loadings play an equally important role, the U-beam structure depicted in Fig. 6 was optimized. The 2000 kg load acting at the tip produces torsion of the beam. As a result the upper flange behaves mainly in flexion, the web, in extension, and the lower flange, both in flexion and extension. For simplicity, only displacement constraints are considered : the tip deflection is limited to 0.1 m, while the relative lateral displacement is limited to 0.005 m (see Fig. 6). Three different methods are employed, which differ by the explicit approximations used for the displacement constraints :

case 1 : full expansion (39), yielding problem (68-70) with $p = 1$ and 3 only (the term in $p = 2$ being zero)

case 2 : linear expansion (25 with $p = 1$), yielding problem (14-16)

case 3 : cubic expansion (25 with $p = 3$).

The iteration history data are given in Table 6 and the final designs in Table 7.

It is interesting to notice that the three methods give similarly good results, which suggests that the use of simple

linear Taylor series expansion in terms of the reciprocal design variables might be a good strategy for most of the structural optimization problems. This idea was previously stated by AUSTIN [23], but not believed valid by many people when he published his paper !

7. CONCLUSIONS AND RECOMMANDATIONS

It is now widely recognized that a powerful and rather general approach to structural optimization is achieved by replacing the original problem with a sequence of explicit approximate problems. This approach was initially conceived for thin walled structures modelled by bar and membrane finite elements, as well in the context of optimality criteria techniques as in the framework of mathematical programming methods using approximation concepts. It has been extended in this work to deal with structural systems made up of beam, plate and flat shell elements, with behavior constraints placed on displacements, stresses, natural frequencies and critical buckling loads. The method presented uses a second order dual algorithm to solve each explicit subproblem. The convergence properties are independent of the number of design variables, which is typical of optimality criteria types of approach as well as linearization techniques in mathematical programming. As a result large structural systems can be treated at the expense of a few finite element analyses.

Several difficult points remain to be clarified. First it is not well known whether the explicit problem can be solved conveniently in any case, because its lack of convexity might lead to discontinuity in the dual function gradient. Secondly, the use of simple linear expansions with respect to the reciprocals of the element transverse sizes, gives rise to satisfactory results in many cases. This suggests that more complicate explicit approximations such as those proposed in this work might not be necessary. Finally, there remains the question of the stress constraints, for which it is difficult to generate adequate first order explicit approximations. Also stress-ratioïng algorithms should be considered for extension-flexion elements, because they correspond to much more inexpensive zero order approximations.

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Table I Iteration History Data for Cantilever Beam

Analysis No.	Linear Expansion Weight (lb)	Linear Expansion Deflection (in)	Cubic Expansion Weight (lb)	Cubic Expansion Deflection (in)
1	1.5000	0.2147	1.5000	0.2147
2	1.0797	0.4996	0.9938	0.6715
3	1.0797	0.5000	1.0660	0.5268
4			1.0760	0.5104
5			1.0764	0.5096
6			1.0766	0.5091
7			1.0767	0.5088
8			1.0768	0.5084
IBM 370-158 CPU time (sec)	92		255	

Table 2 Final Designs for Cantilever Beam

Element No.	Thickness (in)			
	Analytical Ref [20]	Ref [19]	Linear Expansion	Cubic Expansion
1	0.3	0.3004	0.3	0.3
2	0.3	0.3006	0.3	0.3
3	0.3	0.3018	0.3	0.3
4	0.3	0.3056	0.3	0.3
5	0.3	0.3109	0.3	0.3
6	0.3142	0.3167	0.3167	0.3136
7	0.3276	0.3231	0.3287	0.3270
8	0.3396	0.3298	0.3327	0.3389
9	0.3503	0.3367	0.3350	0.3497
10	0.3602	0.3437	0.3379	0.3596
11	0.3693	0.3508	0.3450	0.3687
12	0.3778	0.3579	0.3574	0.3771
13	0.3858	0.3652	0.3711	0.3851
14	0.3933	0.3726	0.3851	0.3926
15	0.4004	0.3803	0.3989	0.3996
16	0.4071	0.3881	0.4121	0.4063
17	0.4135	0.3960	0.4246	0.4126
18	0.4197	0.4040	0.4358	0.4182
19	0.4255	0.4119	0.4446	0.4225
20	0.4312	0.4197	0.4533	0.4268
Weight (lb)	1.0750	1.0453	1.0768	1.0797
No. of Analyses	/	?	8	2

Table 3 Iteration History Data for Square Plate with Deflection Constraint

Analysis No.	Weight (kg)	Deflection (m)
1	9750	0.02029
2	8756	0.001602
3	7799	0.001892
4	7590	0.001958
5	7462	0.001960
6	7358	0.001964
7	7274	0.001980
8	7231	0.001989
9	7206	0.001990
10	7180	0.001989
IBM 370-158 CPU time (sec)	250	

Table 4 : Iteration History Data for Square Plate with Frequency Constraint

Analysis No.	Case 1 : a = 0.1 in		Case 2 : a = 0.05 in		Case 3 : a = 0.001 in	
	Weight (lb)	Frequency (Hz)	Weight (lb)	Frequency (Hz)	Weight (lb)	Frequency (Hz)
1	1.415	19.6	0.849	11.8	1.415	19.5
2	1.097	16.0	0.786	11.5	1.097	16.0
3	0.913	13.6	0.750	11.5	0.902	13.7
4	0.812	12.2	0.708	11.7	0.778	12.4
5	0.771	11.5	0.658	11.6	0.696	11.9
6	0.757	11.3	0.631	11.4	0.647	11.5
7	0.753	11.2	0.617	11.3	0.622	11.4
8	0.752	11.2	0.605	11.3	0.604	11.4
9	0.752	11.2	0.596	11.3	0.588	11.4
10			0.589	11.3	0.574	11.3
11			0.583	11.3	0.559	11.4
12			0.577	11.3	0.542	11.4
13					0.521	11.3
IBM 370-158 CPU time (sec)	218		276		307	

Table 5 : Final Designs for I-beam Structure

thickness (mm) { membrane model / flat shell model

Upper Flange

15.80 16.59	17.30 15.09	11.67 10.08	6.143 5.090	1.815 1.363

Web

Hinged	5.101 2.056	3.602 4.173	3.329 3.965	3.294 3.936	1.997 2.313	Free

↑
Supported

Lower Flange

15.40 20.51	17.01 17.98	11.55 12.21	6.074 6.480	1.792 1.910

Table 6 : Iteration History Data for U-Beam Structure

Analysis No.	Case 1 : Full Expansion			Case 2 : Linear Expansion			Case 3 : Cubic Expansion		
	Weight (kg)	w (m)	$v^+ - v^-$ (m)	weight (kg)	w (m)	$v^+ - v^-$ (m)	weight (kg)	w (m)	$v^+ - v^-$ (m)
1	2340	0.0909	0.00672	2340	0.0909	0.00672	2340	0.0909	0.00672
2	2078	0.0931	0.00507	1894	0.1063	0.00517	2129	0.0937	0.00483
3	1908	0.0976	0.00492	1851	0.0997	0.00563	1933	0.0986	0.00491
4	1842	0.0989	0.00498	1826	0.0997	0.00500	1863	0.0990	0.00496
5	1830	0.0994	0.00499	1817	0.0999	0.00499	1841	0.0995	0.00498
6	1821	0.0995	0.00499	1809	0.0999	0.00499	1831	0.0997	0.00499
7	1813	0.0997	0.00499	1804	0.0999	0.00499	1824	0.0998	0.00499
8	1807	0.0998	0.00499	1799	0.0999	0.00499	1819	0.0999	0.00499
IBM 370-158 CPU time (sec)	335			333			317		

Table 7 : Final Designs for U-Beam Structure

Thickness (mm) { Case 1 : Full Expansion; Case 2 : Linear Expansion; Case 3 : Cubic Expansion }

Upper Flange	21.6	16.6	10.9	6.00	2.29	
	21.2	16.4	11.2	6.57	2.27	
	21.2	16.4	10.8	5.82	2.39	
Web	14.5	11.8	8.61	4.53	27.1	Free End
	14.0	11.1	7.11	3.24	27.0	
	17.0	13.8	9.96	6.00	27.1	
Lower Flange	24.6	22.5	19.4	23.5	17.9	
	25.0	23.4	20.6	23.6	17.9	
	23.0	20.7	18.2	22.8	17.9	

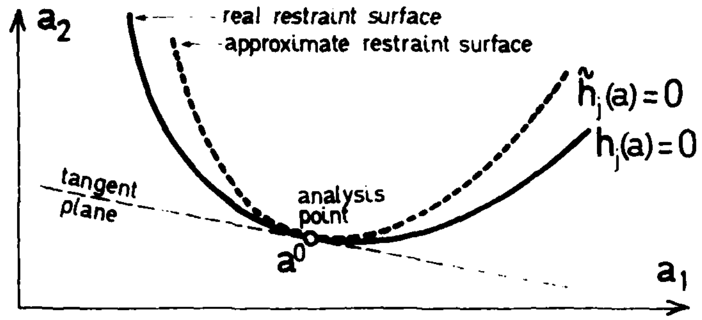


FIG. 1 COMPARISON OF REAL AND APPROXIMATE RESTRAINT SURFACES

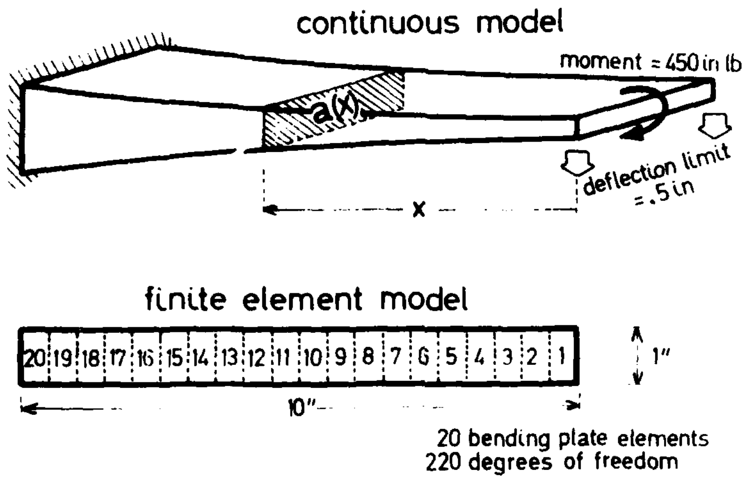
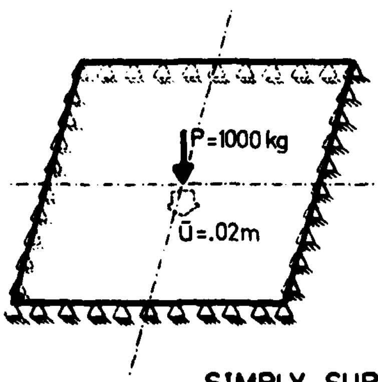


FIG. 2 CANTILEVER BEAM WITH END MOMENT



final design (1/4) thicknesses (m)

symmetrical

.0207	.0356	.0503	.0674	.0932
.0103	.0265	.0409	.0552	.0674
.0223	.0111	.0303	.0409	.0503
.0421	.0229	.0111	.0265	.0356
.0646	.0421	.0223	.0103	.0207

symmetrical

FIG. 3 SIMPLY SUPPORTED SQUARE PLATE WITH DEFLECTION CONSTRAINT

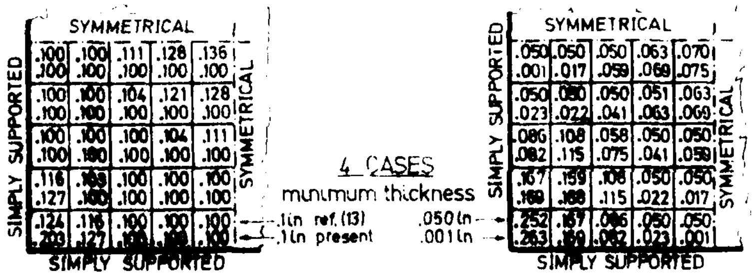


FIG. 4 SIMPLY SUPPORTED SQUARE PLATE WITH FREQUENCY CONSTRAINT

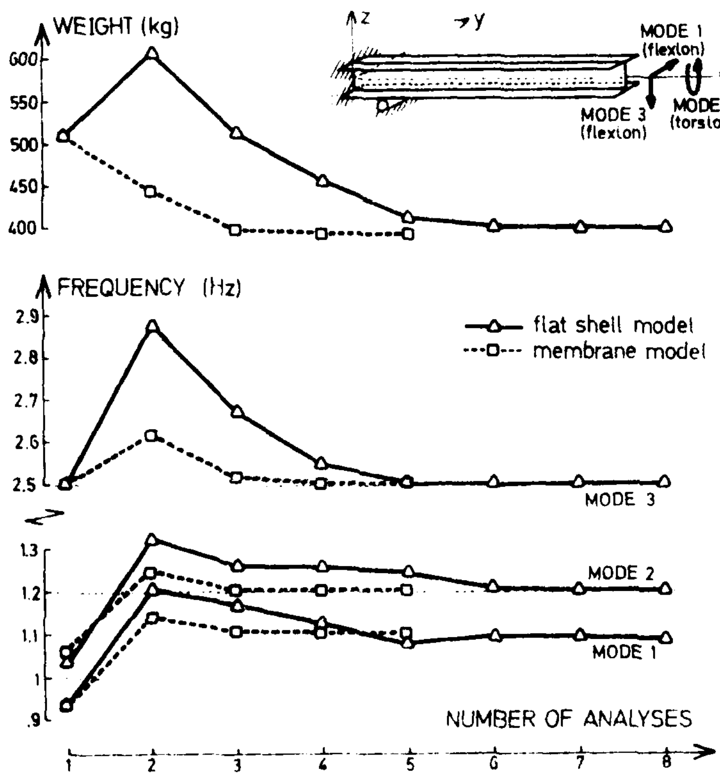


FIG. 5 I-BEAM STRUCTURE

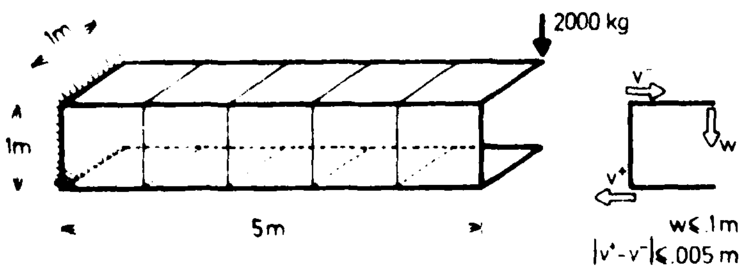


FIG. 6 U-BEAM STRUTURE

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8